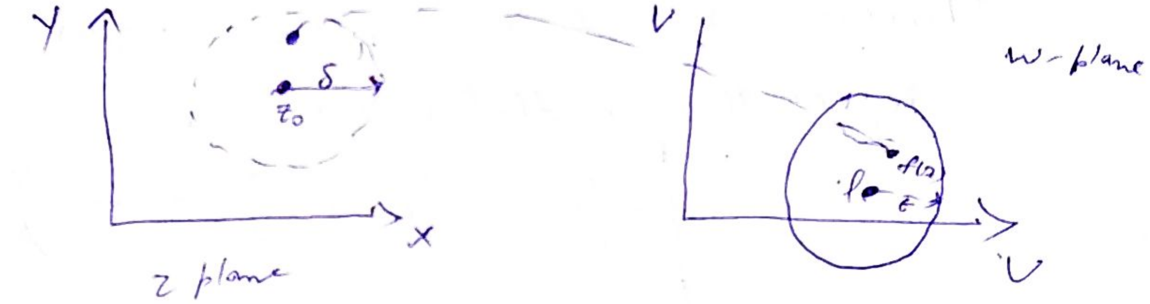
B.Sc Physics Part-III
Mathematical Physics
Paper-V, Group A

Complex function part-II

Limit: A function $w = f(z)$ is said to tend to a limit l as z approaches a point z_0 if for every real ϵ, we can find a positive δ such that

$$|f(z) - l| < \epsilon \quad \text{for } |z - z_0| < \delta ;$$

$$\lim_{z \to z_0} f(z) = l$$



Continuity of f(z):-

f(z) is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Derivative of f(z):-

The derivative of ~~the~~ a single valued function f(z) is defined by

$$\frac{df}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

Subject to the condition that limit exists and has the same value for all the different ways in which δz tends to zero.

Analytic functions: A function $f(z)$ is said to be analytic ~~regular~~ in a region of complex plane if it is single valued and possesses a unique derivative at every point of the region. An analytic function is also called as a regular or holomorphic or a monogenic function. The statement $f(z)$ is analytic at a point $z = z_0$ implies that $f(z)$ has a derivative at every point inside some small circle about $z = z_0$.

Theorem: If $f(z) = u(x, y) + i v(x, y)$ is analytic in a region then in that region

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

The above condition is also called the Cauchy-Riemann (CR) conditions

Example: Find derivative of $f(z) = z^2$.

$$\frac{df}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{(z+\delta z)^2 - z^2}{\delta z} = \lim_{\delta z \to 0} \frac{z^2 + 2z\delta z + (\delta z)^2 - z^2}{\delta z}$$

or $$f'(z) = \lim_{\delta z \to 0} (2z + \delta z) = 2z$$

H.W. (1) Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof.

(2) Use the C.R. conditions to find out whether the following functions are analytic

(i) $y + ix$, (ii) $\frac{y - ix}{x^2+y^2}$, (iii) $\frac{x - iy}{x^2+y^2}$